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ON THE CIRCULATORY SUBSONIC FLOW OF A COMPRESSIBLE FLUID PAST A CIRCULAR CYLINDER

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SUMMARY

The circulatory subsonic flow around an infinite circular cylinder is computed using the linearized pressure-volume relation, by a method developed in a previous report. Formulas and graphs are given for the velocity and pressure distributions, the circulation, the lift, and the dependence of the critical Mach number upon the position of the stagnation point.

INTRODUCTION

The rigorous solution of the differential equations of a two-dimensional steady potential compressible gas flow involves considerable mathematical difficulties. Numerical integration is rather laborious and hardly ever yields results of a general character. Therefore, considerable attention was given to approximate analytical methods.

Tchaplygin (reference 1) noticed that the differential equation of the velocity potential takes a rather simple form if the exponent γ in the polytropic pressure-density relation is replaced by -1. The equation then becomes the well-known equation of a minimal surface. Furthermore, in the so-called hodograph plane, the equation may be transformed into the Laplace equation. The physical meaning of the linearized pressure-volume relation ($\gamma = -1$) has been clarified by Busemann (reference 2) and especially by Von Kármán (reference 3) and Tsien (reference 4). They showed that using this relation amounts to replacing the actual pressure-volume curve by its tangent. Another way of justifying the use of the linearized equation of state is indicated in this report.

Tsien derived a formula transforming a given incompressible flow around a closed profile into a compressible flow (satisfying the linearized equation of state) around another (slightly distorted) profile. However, this formula is applicable only to circulation-free flows. In a previous report (reference 5) the author developed formulas which permit the construction of circulatory flows as well. It has been shown that under very general conditions every incompressible flow yields a compressible flow and that <u>all</u> compressible flows may be obtained in this way.

In the present report, this method is tested and illustrated by computing the circulatory flow around a circular cylinder. (The circulation-free flow around a circular cylinder has been treated by Tamada, who used Tsien's method.) (See reference 6.) The circular profile, although of no technical importance, has been chosen because of the simplicity of the computations and the possibility of comparing results with those obtained by different methods.

It should be emphasized that the use of the linearized pressure-volume relation restricts the investigation to purely subsonic flows.

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SYMBOLS

a local speed of sound

a_o speed of sound at a stagnation point

C_L lift coefficient

$G(\zeta)$ normalized complex potential of an incompressible flow in the ζ-plane

L lift per unit span

M local Mach number

M_∞ stream Mach number (Mach number at infinity)

$M_{\infty,c}$ critical stream Mach number

n parameter defined by the equation $n = \sqrt{1 + (q_\infty/a_0)^2}$

P profile in the z-plane (the plane of the compressible flow)

p pressure

p_0 pressure at the stagnation point

q local speed of a compressible flow

q_s value of q for which $q = a$

q_∞ speed of the compressible flow at infinity

R radius of the circular cylinder

R_1 auxiliary parameter defined by the equation $R_1 = \cos\delta/\cos\beta$

R_P radius of curvature of the profile P

R_Γ radius of curvature of the profile Γ

r parameter defined by the equation $2r/(1 - r^2) = q_s/a_0$

S pressure coefficient

u,v components of the local velocity of the compressible flow

$z = x + iy$ complex variable in the plane of the compressible flow

Z_1,Z_2,Z_3 auxiliary complex variables

α absolute value of the argument of a stagnation point of the compressible flow about a circular cylinder (angle of attack)

$\beta,\delta,\beta_1,\beta_2$ angles used to locate stagnation points of auxiliary flows in the ζ-plane

γ exponent in the polytropic relation

Γ_c circulation of the compressible flow

Γ_i circulation of the incompressible flow

ϵ angle defined by the equation $\delta = \pi/2 - \epsilon$

$\zeta = \xi + i\eta$ complex variable in the plane of the auxiliary incompressible flow

Λ argument of a point on the circle $|Z| = R$

μ parameter defined by the equation $\mu^2 = (n - 1)/(n + 1)$

Π profile in the plane of the incompressible flow (ζ-plane)

ρ density

ρ_o stagnation density

σ absolute value of the argument of a stagnation point of the flow in the Z-plane

Φ, Ψ angles used to locate points on the profile Π

φ velocity potential of the compressible flow

χ angle defined by the equation $\chi = (\pi/2 - \delta)/(2 - n)$

ω argument of a point on the cross section of the circular cylinder

ANALYSIS

1. Reduction of the Problem to the Differential Equation of a Minimal Surface

Let u and v denote the x and y components of the velocity of a steady two-dimensional flow of a compressible fluid. If the flow is irrotational, a velocity potential $\varphi(x,y)$ may be introduced, such that

$$u = a_o \frac{\partial \varphi}{\partial x} \qquad v = a_o \frac{\partial \varphi}{\partial y} \tag{1}$$

where a_o is the speed of sound at a stagnation point. The continuity equation yields the equation

$$\frac{\partial}{\partial x}\left(\rho \frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho \frac{\partial \varphi}{\partial y}\right) = 0 \tag{2}$$

ρ being the density.

If the flow occupies the domain exterior to a closed profile P and at infinity approaches a uniform flow in the positive x direction, φ must satisfy the boundary conditions

$$\frac{\partial \varphi}{\partial x} \rightarrow \frac{q_\infty}{a_o}, \quad \frac{\partial \varphi}{\partial y} \rightarrow 0 \quad \text{as} \quad x^2 + y^2 \rightarrow \infty \tag{3}$$

where q_∞ is the speed at infinity (speed of the undisturbed flow), and

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{on} \quad P \tag{4}$$

$\left(\frac{\partial}{\partial n}\right.$ denotes differentiation in the direction normal to the profile $\left.P\right)$.

The speed of the fluid q is given by

$$q^2 = u^2 + v^2 = a_o^2\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right]$$

and the Kutta-Joukowski condition requires that

$$q < \infty \tag{5}$$

It will be assumed that the pressure p and the density are connected by the relation

$$p = A + B\rho^\gamma \tag{6}$$

where A, B, and γ are constants. For all physically important cases

$$1 \leq \gamma \leq 2$$

The standard value of γ for air is 1.405. Equation (6) and Bernoulli's theorem imply that

$$\rho = \rho_o\left(1 - \frac{\gamma - 1}{2}\frac{q^2}{a_o^2}\right)^{\frac{1}{\gamma-1}} \tag{7}$$

$$a^2 = a_o^2 - \frac{\gamma - 1}{2} q^2 \tag{8}$$

$$M^2 = \frac{(q/a_o)^2}{1 - \frac{\gamma - 1}{2}(q/a_o)^2} \tag{9}$$

Here ρ_o is the density at a stagnation point, $a = dp/d\rho$, the local speed of sound and $M = q/a$, the local Mach number. It is seen that the flow is subsonic as long as

$$q < q_s = a_o\sqrt{\frac{2}{\gamma + 1}} \tag{10}$$

Equation (7) can be written as a relation (depending on the parameter γ) between the two dimensionless quantities ρ/ρ_o and q/a_o:

$$\frac{\rho}{\rho_o} = f\left(\frac{q}{a_o}, \gamma\right) \tag{11}$$

Equation (2) can now be written as

$$\frac{\partial}{\partial x}\left\{f\left[\sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2}, \gamma\right]\frac{\partial\varphi}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{f\left[\sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2}, \gamma\right]\frac{\partial\varphi}{\partial y}\right\} = 0 \tag{2'}$$

Expanding the right side of (11) into a power series in q/a_o yields

$$f\left(\frac{q}{a_o},\ \gamma\right) = 1 - \frac{1}{2}\left(\frac{q}{a_o}\right)^2 + \frac{2 - \gamma}{8}\left(\frac{q}{a_o}\right)^4 - + \ . \ . \ .$$

Evidently, for small values of q/a_o the relation between ρ and q/a_o is only slightly affected by the value of γ. In particular, if the actual value of γ is replaced by -1, the resulting error in ρ/ρ_o is of the same order of magnitude as

$$\frac{\gamma + 1}{8}\left(\frac{q}{a_o}\right)^4$$

Table I gives the values of ρ/ρ_o for $\gamma = 1.405$ and $\gamma = -1$ and for values of q/a_o from 0 to 0.912 (0.912 is the value of q_s/a_o for $\gamma = 1.405$).

On the basis of the foregoing remarks it may be concluded that for flows where q/a_o is small only a slight error will be made if the actual value of γ is replaced by -1 in (11). Since

$$f\left(\frac{q}{a_o},\ -1\right) = \frac{1}{\sqrt{1 + (q/a_o)^2}}$$

TABLE I.- DENSITY AS A FUNCTION OF SPEED

$\frac{q}{a_o}$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.912
$\gamma = 1.405$											
$\frac{q}{\rho_o}$	1.000	0.995	0.980	0.956	0.922	0.880	0.830	0.773	0.710	0.643	0.634
$\gamma = -1.000$											
$\frac{\rho}{\rho_o}$	1.000	0.995	0.980	0.958	0.929	0.894	0.858	0.820	0.780	0.743	0.739

equation (2') becomes

$$\left\{1 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right\} \frac{\partial^2\varphi}{\partial x^2} - 2 \frac{\partial\varphi}{\partial x} \frac{\partial\varphi}{\partial y} \frac{\partial^2 \varphi}{\partial x\partial y} + \left\{1 + \left(\frac{\partial\varphi}{\partial x}\right)^2\right\} \frac{\partial^2\varphi}{\partial y^2} = 0 \quad (12)$$

This is the well-known equation of a minimal surface, and its solution can be expressed in terms of analytic functions of a complex variable.

After the values of q/a_o have been found by integrating the approximate equation (12) under the boundary conditions (3), (4), and (5), the values of the density and the local Mach number must be determined by means of formulas (7) and (9) with the correct value of γ.

The use of the approximate equation (12) can also be justified by the remark that the constants A and B can be determined so that the curve (in the p, ρ-plane) given by the linear pressure-volume relation

$$p = A + \frac{B}{\rho} \quad (13)$$

will be at some point tangent to the curve given by the actual pressure-density relation

$$p\rho^{-\gamma} = p_o \rho_o^{-\gamma}$$

(see references 3 and 4). The linear pressure-volume relation was introduced by Tchaplygin and has been used by Busemann, Demtchenko, Von Kármán, and Tsien.

2. Formulas for the Solution of the Approximate Problem

In a previous report (see reference 5) a method was given which permits the construction of solutions of equation (12) satisfying the boundary conditions (3), (4), and (5). It has been shown that this method yields solutions for all profiles P possessing at most two sharp edges. The only restrictive condition concerns the value of q_∞/a_o which should not exceed $\sqrt{\frac{3}{4}} = 0.866$. Now for $\gamma = 1.405$, $q_s = 0.912$. Thus, the method would fail only for profiles for which the maximum local speed is very close to the speed of the undisturbed flow - that is, for very thin profiles.

This method is based upon a transformation of an incompressible flow into a solution of equation (12).

Let Π be a closed profile in the ζ-plane, $(\zeta = \xi + i\eta)$ possessing at most two edges. Let $G(\zeta)$ be a normalized complex potential (i.e., the function G = [velocity potential + i(stream function)] such that the speed at infinity equals 1) of an incompressible flow around the profile Π; $G(\zeta)$ is an analytic function defined in the domain exterior to Π and satisfies the conditions

$$G'(\infty) = 1$$

$$\operatorname{Im} G = \text{constant on } \Pi$$

as well as the Kutta-Joukowsky condition

$$|G'|_{max} < \infty$$

(Im = imaginary part of). Let n be a real number such that

$$1 < n < 2$$

The function

$$(2.1)\; z = x + iy = \frac{n+1}{2}\int G'(\zeta)^{1-\frac{1}{n}}\, d\zeta - \frac{n-1}{2}\overline{\int G'(\zeta)^{1+\frac{1}{n}}\, d\zeta} \qquad (14)$$

(the bar denotes the conjugate complex quantity) maps the domain exterior to Π into a domain exterior to a closed profile P. The function

$$\varphi = \sqrt{n^2 - 1}\ \operatorname{Re} G \qquad (15)$$

(Re = real part of) considered as a function of x and y satisfies the differential equation (12) and the boundary conditions, (4), (5), and (6), where

$$\frac{q_\infty}{a_0} = \sqrt{n^2 - 1} \qquad (16)$$

Thus φ may be considered as the potential of a compressible flow around P. The speed q of the compressible flow (at a point z corresponding to a point ζ) is given by

$$\frac{q}{a_o} = \frac{2\left(\frac{n-1}{n+1}\right)^{\frac{1}{2}} |G'(\zeta)|^{1/n}}{1 - \left(\frac{n-1}{n+1}\right)|G'(\zeta)|^{2/n}}$$

$$= \frac{2q_\infty/a_o\ |G'(\zeta)|^{1/n}}{n + 1 - (n - 1)|G'(\zeta)|^{2/n}} \tag{17}$$

The angle between the direction of the compressible flow and the x-axis,

$$\theta = \tan^{-1}\frac{v}{u}$$

is given by

$$\theta = \frac{1}{n}\arg G'(\zeta) \tag{18}$$

(arg = argument of). In particular, the slope of the profile P is 1/n times the slope of the profile Π, at a corresponding point. If R_P denotes the radius of curvature of P at a point z and R_Π the radius of curvature at the corresponding point ζ of Π, then

$$R_P = n\left\{\frac{n+1}{2}\left|G'(\zeta)\right|^{1-1/n} - \frac{n-1}{2}\left|G'(\zeta)\right|^{1+1/n}\right\} R_\Pi \tag{19}$$

The profile P can be constructed graphically using this information.

The proofs of these assertions will be found in the report quoted above.

Note that for values of n which are close to 1 (i.e., for small values of q_∞/a_o) the profile P will be slightly different from Π. For, then, the coefficient of the first integral in (14) is close to 1, the exponent of the integrand is close to 0, so that the first term on the right side of (14) is close to ζ, and the second term is small compared to the first.

However, the slopes of P and Π (at corresponding points) are different. If it is desired that the profile P should possess at the two stagnation points angles β_1 and β_2, the profile $\sqcap$ must possess at its stagnation points the angles $n\beta_1$ and $n\beta_2$.

It will be noted that the method does not yet permit the solution of the boundary value problem for a given profile. However, it is possible to choose the "conjugate profile" $\sqcap$ in such a manner that the profile P will differ very little from a given profile.

Remark: If G is a one-valued function (i.e., for a flow without circulation), the formulas (14), (15), (17), (18), and (19) can be replaced by the simpler relations which are equivalent to the formulas found by Tsien (see reference 4)

$$z = \frac{n+1}{2}\zeta - \frac{n-1}{2}\overline{\int G'(\zeta)^2\, d\zeta} \tag{20}$$

$$\varphi = \sqrt{n^2 - 1}\ \operatorname{Re} G \tag{21}$$

$$q = \frac{2\,\frac{n-1}{n+1}|G'(\zeta)|}{1 - \left(\frac{n-1}{n+1}\right)^2 |G'(\zeta)|^2} \tag{22}$$

$$\theta = \tan^{-1} G'(\zeta) \tag{23}$$

$$R_P = \left\{\frac{n+1}{2} - \frac{n-1}{2}\,|G'(\zeta)|^2\right\} R_\sqcap \tag{24}$$

However, if there is circulation, formula (20) does not yield a closed profile.

3. Incompressible Flow Yielding a Compressible Flow around a Nearly Circular Cylinder

The method described in the foregoing will be applied presently to the construction of a circulatory compressible flow around a nearly circular profile P. It is natural to start with a profile $\sqcap$ consisting of an upper arc of the unit circle $|\zeta| = 1$ and a lower arc of a circle which intersects the first arc at the points $\zeta = e^{-i\delta}$ and $\zeta = -e^{i\delta}$ and forms the angles $n\pi$ with the first arc. (See fig. 1.) The center of the lower arc is situated at $\zeta = i(R_1 \sin \beta - \sin \delta)$ where

$$\beta = \delta - (n - 1)\pi \tag{25}$$

Its radius is equal to

$$R_1 = \cos \delta/\cos \beta$$

It may be assumed without loss of generality that $0 < \delta < \frac{\pi}{2}$ and $R_1 < 1$.

Let $G(\zeta)$ be the complex potential of an incompressible flow past $\sqcap$ which possesses stagnation points at $\zeta = e^{-i\delta}$ and $\zeta = -e^{i\delta}$ and has the velocity 1 at infinity.

In order to compute $G(\zeta)$ the domain exterior to $\sqcap$ is mapped conformally into the domain $|Z| > R$ in an auxiliary Z-plane, by a transformation which takes $\zeta = \infty$ into $Z = \infty$ and satisfies the condition $dZ/d\zeta > 0$ at infinity. The points $\zeta = e^{-i\delta}$ and $\zeta = -e^{i\delta}$ are taken into the points $Z = Re^{-i\sigma}$ and $Z = -Re^{i\sigma}$, respectively, where

$$\sigma = \frac{\delta}{2 - n} - \frac{n - 1}{2 - n}\frac{\pi}{2} \tag{27}$$

Set

$$R = \frac{\cos \delta}{(2 - n)\cos \sigma} \tag{28}$$

then $(dZ/d\zeta)_{\zeta=\infty} = 1$. Hence $G(\zeta)$, considered as a function of Z, is given by

$$G = Z + \frac{R^2}{Z} + 2\, Ri \sin \sigma \log Z \tag{29}$$

At a point $\zeta = e^{i\Phi}$ of the upper arc of $\sqcap$ the speed of the incompressible fluid is given by

$$G'(e^{i\Phi}) = 2R^2 \frac{|\sin \Lambda + \sin \sigma|^2}{|\sin \phi + \sin \delta|} \tag{30}$$

At a point $\zeta = i(R_1 \sin \beta - \sin \delta) + R_1 e^{i\psi}$ of the lower arc of $\sqcap$ the speed of the incompressible flow is given by

$$\left| G'\left[i(R_1 \sin \beta - \sin \delta) + R_1 e^{i\psi} \right] \right| = 2 \frac{R^2}{R_1^2} \frac{|\sin \Lambda + \sin \sigma|^2}{|\sin \psi + \sin \beta|} \tag{31}$$

In these formulas Λ is the argument of the point on the circle $|Z| = R$ into which the point ζ is taken. The angle $\frac{1}{2}(\Lambda - \sigma)$ can be determined by the formulas

$$\tan \frac{1}{2}(\Lambda - \sigma) = \frac{1}{\cos \sigma} \left\{ \left| \frac{\sin \frac{1}{2}(\phi + \delta)}{\cos \frac{1}{2}(\phi - \delta)} \right|^{\frac{1}{2-n}} - \sin \sigma \right\} \tag{32}$$

(for a point on the upper arc) and

$$\tan \frac{1}{2}(\Lambda - \sigma) = - \frac{1}{\cos \sigma} \left\{ \left| \frac{\sin \frac{1}{2}(\psi + \beta)}{\cos \frac{1}{2}(\psi - \beta)} \right|^{\frac{1}{2-n}} + \sin \sigma \right\} \tag{33}$$

(for a point on the lower arc). The details of the computation will be found in the appendix.

The maximum of $|G'|$ is reached at the top of the profile $\sqcap$, where $\Phi = \Lambda = \frac{\pi}{2}$. Hence

$$|G'|_{max} = 2(1 + \sin \delta)R^2 \frac{\cos^4 \frac{1}{4}(\pi - 2\sigma)}{\cos^4 \frac{1}{4}(\pi - 2\delta)}$$

This is easily transformed into

$$|G'|_{max} = \frac{2}{(2 - n)^2} (1 - \sin \delta) \cot^2 \frac{\pi/2 - \delta}{2(2 - n)} \tag{34}$$

The incompressible flow around $\sqcap$ is now transformed into a compressible flow around the profile P (in the z-plane). The transformation is given by the formulas of the preceding section. By virtue of (17) and the actual equation of state, the maximal local Mach number will be equal to 1 if

$$|G'|_{max} = \left(\sqrt{\frac{n + 1}{n - 1}}\, r\right)^n$$

where r is determined from the equation

$$\frac{2r}{1 - r^2} = q_s/a_o = \sqrt{\frac{2}{\gamma + 1}}$$

For $\gamma = 1.405$

$$r = 0.3875 \ldots$$

Thus, for a given δ, the maximal admissible n will be given by the relation

$$\frac{2}{(2 - n)^2}(1 - \sin\delta)\cot^2\frac{\pi/2 - \delta}{2(2 - n)} = \left(\frac{n + 1}{n - 1}\right)^{n/2} r^n$$

Set

$$\delta = \frac{\pi}{2} - 2\epsilon \tag{35}$$

then ϵ is to be determined as the positive root of the equation

$$A(n)\sin\epsilon = \tan\frac{\epsilon}{2 - n} \tag{36}$$

with

$$A(n) = \frac{2r^{-n/2}}{2 - n}\left(\frac{n - 1}{n + 1}\right)^{n/4} \tag{37}$$

Numerical values of n and ϵ determined by this equation (for $\gamma = 1.405$) are given in table II. It is seen that the values of n are very close to 1.

The radius of curvature of the profile P can be computed by means of formula (19). The result of this computation for various values of n and δ are given in table III. It will be noted that the radius of curvature of P is nearly constant. Therefore, the profile P is nearly a circle of unit radius. A point $\zeta = e^{i\Phi}$ of the upper arc is taken approximately into the point $e^{i\omega}$ with

$$\omega = \frac{\Phi}{n} + \frac{n - 1}{n}\frac{\pi}{2} \tag{38}$$

A point $\zeta = i(R_1 \sin \beta - \sin \delta) + R_1 e^{i\psi}$ of the lower circle is taken approximately into the point $e^{i\omega}$ with

$$\omega = \frac{\psi}{n} + \frac{n - 1}{n}\frac{\pi}{2} \tag{39}$$

This follows from the fact that the slope of P at a point z must equal $1/n$ times the slope of $\sqcap$ at a corresponding point ζ. In particular, the stagnation points of the compressible flow will be situated at $e^{-i\alpha}$ and $-e^{i\alpha}$, where

$$\alpha = \frac{\delta}{n} - \frac{n - 1}{n}\frac{\pi}{2} \tag{40}$$

α will be positive for $\delta > (n - 1)\pi/2$.

Greater accuracy could be achieved by taking as the radius of P the arithmetic mean of the values of R_p, or to obtain this radius by a graphical construction of P. However, this correction seems to be too insignificant to justify the additional computational labor. Of course, the size of P is of no importance as far as the velocity distribution is concerned.

The graphical construction of the profile P is shown in figure 2. Figure 3 contains several examples of profiles P constructed by this method. The deviation of P from a circle is remarkably small.

4. Critical Mach Number

The critical stream Mach number $M_{\infty,c}$ is defined as the value of M_∞ for which the maximal local Mach number equals 1. In order to compute the critical Mach number for a flow around a circle with the angle of attack α (the angle of attack being defined as the negative argument of a stagnation point), set

$$\alpha = \frac{2 - n}{n}\frac{\pi}{2} - \frac{2\epsilon}{n} \tag{41}$$

and

$$M_{\infty,c}^2 = \frac{n^2 - 1}{1 - \frac{1}{2}(\gamma - 1)(n^2 - 1)} \tag{42}$$

where n and ϵ are connected by equation (36). [Cf. (9), (16), and (40).] In this way the values of $M_{\infty,c}$ as a function of α (for $\gamma = 1.405$) given in table IV have been computed. The relation between $M_{\infty,c}$ and α is plotted in figure 4.

5. Velocity and Pressure Distribution

In order to compute the velocity distribution of a compressible flow along a circular profile, set

$$n^2 = 1 + \frac{M_\infty^2}{1 + \frac{\gamma - 1}{2} M_\infty^2} \tag{43}$$

$$\delta = n\alpha + (n - 1)\frac{\pi}{2} \tag{44}$$

where M_∞ is the desired stream Mach number and α the desired angle of attack. Then the dimensionless speed of the compressible flow at a point $e^{i\omega}$ of the circle is given by formula (17). And $|G'|$ must be computed by means of formulas (30) and (31). The first is to be used for $\frac{\pi}{2} \geq \omega > -\alpha$, the second for $-\frac{\pi}{2} \leq \omega \leq -\alpha$. (Note that the velocity distribution is symmetrical with respect to the vertical axis.) The values of Φ and ψ corresponding to a given value of Λ are given by equations (38) and (39). In this way the velocity distribution plotted in figure 5 has been computed.

Once the velocity distribution is known, the pressure distribution may be determined by means of equations (7) and the actual equation of state. The dimensionless pressure coefficient

$$S = \frac{p_o - p}{\frac{1}{2}\rho_\infty q_\infty^2}$$

is given by

$$S = \frac{2}{\gamma} \frac{1 - \left(1 - \frac{\gamma - 1}{2}\frac{q^2}{a_o^2}\right)^{\gamma/(\gamma-1)}}{\frac{q_\infty^2}{a_o^2}\left(1 - \frac{\gamma - 1}{2}\frac{q_\infty^2}{a_o^2}\right)^{\frac{1}{\gamma-1}}} \tag{45}$$

For an incompressible flow

$$S = \left(\frac{q}{q_\infty}\right)^2. \tag{46}$$

The values of S plotted in figure 6 have been computed by these formulas.

6. Circulation and Lift

By virtue of (1) the circulation of the compressible flow is equal to

$$\Gamma_c = \oint u\,dx + v\,dy = a_o \oint d\varphi$$

where the integration is extended over a closed curve around the profile P. By (15)

$$\Gamma_c = a_o\sqrt{n^2 - 1}\oint G'(\zeta)d\zeta$$

integrating over a closed curve around Π. But by (29)

$$\oint G'(\zeta)d\zeta = -4\pi R \sin\sigma$$

Introduce the values of R, σ, and δ given by (27), (28), and (40) to obtain the equation

$$\Gamma_c = -4\pi \frac{\sqrt{n^2 - 1}}{2 - n} a_o \cos\left\{n\alpha + (n - 1)\frac{\pi}{2}\right\} \tan \frac{n\alpha}{2 - n}$$

or, by (16)

$$\Gamma_c = -4\pi q_\infty \frac{1}{2 - n} \cos\left\{n\alpha + (n - 1)\frac{\pi}{2}\right\} \tan \frac{n\alpha}{2 - n} \qquad (47)$$

n is given in terms of M_∞ by formula (43).

For an incompressible flow with the same velocity at infinity and the same position of a stagnation point the circulation is given by

$$\Gamma_i = -4\pi q_\infty \sin \alpha$$

The values of Γ_c/Γ_i are given in table V and plotted in figure 7. It will be seen that compressibility results in a larger circulation. The additional circulation due to compressibility increases as the Mach number increases but it decreases as the angle of attack increases.

The Kutta-Joukowsky lift formula holds also for compressible flow. (See reference 7.) The lift (per unit span) is given by

$$L = \rho_\infty q_\infty |\Gamma_c|$$

The dimensionless lift coefficient for a circle

$$C_L = \frac{L}{\frac{1}{2}\rho_\infty q_\infty^2 r}$$

(r being the radius of the circle) is obtained from (47) and (16), noticing that this last formula holds for $r = 1$. Thus

$$C_L = \frac{8\pi}{2 - n} \cos\left\{n\alpha + (n - 1)\frac{\pi}{2}\right\} \tan \frac{n\alpha}{2 - n} \qquad (48)$$

Here α is the angle of attack and n is determined by the stream Mach number. (See equation (43).) For an incompressible fluid

$$C_L = 8\pi \sin \alpha \tag{49}$$

The values of C_L are given in table VI and plotted in figure 8.

7. Comparison with the Kármán-Tsien Method

The application of the linearized pressure-density relation $(\gamma = -1)$ to compressible flow problems consists of two steps: (1) the solution of a boundary value problem for the equation of the minimal surface, and (2) the interpretation of the results. The first step is purely mathematical. Concerning the second, the following point of view has been adopted in the present report. The velocity field (i.e., the values of q/a_0) computed under the assumption $\gamma = -1$ is considered as an approximation to the velocity field of a flow satisfying the actual equation of state (with $\gamma = 1.405$) which has the same velocity at infinity. Accordingly, the stream Mach number M_∞ is computed from q_∞/a_0 by means of the formula (9) obtained from the actual equation of state. Von Kármán and Tsien (references 3 and 4) adopted a different point of view. They consider the fictitious flow satisfying the linearized equation of state as an approximation to the flow of a compressible fluid possessing the same stream Mach number. Consequently, they compute the speed of the undisturbed flow q_∞ from the stream Mach number by means of the formula

$$q_\infty^2 = a_0^2 \frac{M_\infty^2}{1 - M_\infty^2} \tag{50}$$

which follows from the linearized equation of state. (Furthermore, they take over from the results obtained by setting $\gamma = -1$ only the values of q/q_∞.) Other interpretations of the results are also possible. Their relative merits can be determined only by comparison with rigorous solutions. If equation (50) is compared with equation (9), it is seen that the theoretical and numerical results of the present report can be adapted to the Von Kármán-Tsien point of view by replacing the value M_∞ wherever it occurs by the value

$$M_\infty(\text{Kármán-Tsien}) = \frac{M_\infty}{\sqrt{1 + \frac{\gamma + 1}{2} M_\infty^2}} \tag{51}$$

This relation is plotted in figure 9b. It is seen that for the values of the stream Mach number occurring in flows around a circular cylinder the difference between the two values of M_∞ is quite small.

For the case of a circulation-free flow equation (22) leads to the following velocity correction formula

$$\frac{q}{q_\infty} = \frac{q_i}{q_{i,\infty}} \frac{1 - \mu^2}{1 - \mu^2(q_i/q_{i,\infty})} \tag{52}$$

Here

$$\mu^2 = \frac{n - 1}{n + 1}$$

so that, by (43)

$$\mu = \frac{M_\infty}{\sqrt{1 + \frac{\gamma + 1}{2} M_\infty^2} + \sqrt{1 + \frac{\gamma - 1}{2} M_\infty^2}} \tag{53}$$

q_i is the speed of the incompressible flow around the profile Π (i.e., of the flow with the complex potential G). If profile distortion (the difference between the profiles Π and P) is neglected, q_i may be taken as the speed of an incompressible flow around P. Formula (52) is the Von Kármán-Tsien velocity correction formula, except that these authors obtain for μ the value

$$\mu = \frac{M_\infty}{1 + \sqrt{1 - M_\infty^2}}$$

in accordance with their method of interpreting results obtained by setting $\gamma = -1$. (The values at μ^2 are plotted in figure 9a.)

Theoretically, formula (52) may be employed for circulation-free flows only. Nevertheless, it seems worth while to try to use it for circulatory flows as well. Velocity distributions computed by means of (52) are shown in figure 5. Figure 4 gives the values of the critical stream Mach number computed by means of (52). The agreement with results obtained by the method of this report is rather surprising. It is due to the fact that for circulatory flows around a circular profile, the critical values of the stream Mach number are very low. Greater discrepancy should be expected in the case of slender profiles.

Remark: The Glauert-Prandtl correction formula is derived under the assumption of a nearly uniform flow and should not be used for a circular profile.

8. Comparison with the Method of Successive Approximations

The method of successive approximations has been applied to the circulatory flow past a circular cylinder. The first approximation to the velocity potential has been computed by Lamb (reference 8), who used the Rayleigh-Janzen scheme and by Tamatiko and Umemato (reference 9), who used the Poggi method. In figure 5 velocity distributions computed according to the formula given by Tamatiko and Umemato are plotted. Figure 4 shows the values of the stream Mach number given by these authors. Tamatiko and Umemato also give a formula for the circulation

$$\frac{\Gamma}{\Gamma_i} = 1 + \left(\frac{11}{12} + \frac{1}{3}\sin^2\alpha\right)M_\infty{}^2$$

This formula leads to results quite different from those obtained in the present report.

The second approximation to the circulatory flow around a circular cylinder has been computed recently by Heaslit. (See reference 10.) His numerical results are given in a form which does not permit an immediate comparison with the ones given in this report.

CONCLUDING REMARKS

It has been shown that the formula, transforming a circulatory incompressible flow around a closed profile into a compressible flow (obeying the linearized equation of state) may be used for the effective approximate computation of a purely subsonic flow around a profile closely approximating a given shape.

The following qualitative results have been obtained for the case of a circular cylinder. It may be expected that they hold also for other shapes.

1. The critical value of the stream Mach number $M_{\infty,c}$ decreases as the angle of attack α (for the case of a circle: the argument of the stagnation point) increases. However, $]dM_{\infty,c}/d\alpha[$ decreases as α increases.

2. Compressibility results in a higher value of the circulation (and lift) than the one predicted by the theory of incompressible fluids for the same position of the stagnation points. This effect increases as the stream Mach number increases. However, it becomes less pronounced as the angle of attack increases.

3. The Von Kármán-Tsien velocity correction formula (which is theoretically applicable only for flows without circulation) yields good approximate results for circulatory flows of small stream Mach number and small angle of attack.

Brown University,
Providence, R. I., September 1, 1944.

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APPENDIX

THE INCOMPRESSIBLE FLOW AROUND THE PROFILE ⊓

The transformation

$$Z_1 = \frac{\zeta + e^{i\delta}}{\zeta - e^{-i\delta}} \tag{A1}$$

takes the domain exterior to the profile ⊓ (see fig. 1) into a sector bounded by two rays through the origin which make the angles $(\pi/2 + \beta)$ and $-(\pi/2 - \delta)$ with the real axis. The transformation

$$Z_2 = Z_1 \, e^{(\pi/2 - \delta)i} \tag{A2}$$

rotates this sector so that the lower ray coincides with the real axis. The transformation

$$Z_3 = Z_2^{1/(2-n)} \tag{A3}$$

maps the rotated sector into the upper half-plane. The point $\zeta = \infty$ is taken into $Z_1 = 1$ by the first transformation, into $Z_2 = e^{(\pi/2-\delta)i}$ by the second and into $Z_3 = e^{i\chi}$ by the third, where

$$\chi = \frac{\pi/2 - \delta}{2 - n}$$

Finally the transformation

$$Z = Re^{-i\sigma}\,\frac{Z_3 - e^{-i\chi}}{Z_3 - e^{i\chi}} \tag{A4}$$

(R, σ real constants) takes the upper half-plane of the Z_3 plane into the domain $|Z| > R$. The point $Z_3 = e^{i\chi}$ (i.e., the point $\zeta = \infty$) is taken into $Z = \infty$. The two stagnation points, $\zeta = e^{-i\delta}$ and $\zeta = -e^{i\delta}$ are taken into $Z = Re^{-i\sigma}$ and $Z = -Re^{i\sigma}$, respectively.

Next, the constants R and σ must be determined so that

$$\lim_{\zeta \to \infty}\left(\frac{dZ}{d\zeta}\right) = \lim_{\zeta \to \infty}\left(\frac{dZ}{dZ_3}\,\frac{dZ_3}{dZ_2}\,\frac{dZ_2}{dZ_1}\,\frac{dZ_1}{d\zeta}\right) = 1$$

A simple computation yields the values

$$R = \frac{\cos\delta}{(2 - n)\cos\sigma}, \qquad \sigma = \frac{\pi}{2} - \chi$$

Thus, formulas (27) and (28) are verified.

It follows that $G(\zeta)$ considered as a function of Z must have the form (3.5).

In order to compute the correspondence between the points of the profile $\sqcap$ and those of the circle $|Z| = R$, note that by (A1) to (A4)

$$\frac{\zeta + e^{i\delta}}{\zeta - e^{-i\delta}} = \left(\frac{Z + Re^{i\sigma}}{Z - Re^{-i\sigma}}\right)^{2-n} \qquad (A5)$$

Equation (A5) is that of a Von Kármán-Trefftz transformation, as should have been expected. Let $\zeta = e^{i\Phi}$ be a point on the upper circle of the profile $\sqcap$ and $Z = Re^{i\wedge}$ the corresponding point on $|Z| = R$. Substitute in (A5) and take absolute values:

$$\left|\frac{\cos\frac{1}{2}(\Phi - \delta)}{\sin\frac{1}{2}(\Phi + \delta)}\right| = \left|\frac{\cos\frac{1}{2}(\wedge - \sigma)}{\sin\frac{1}{2}(\wedge + \sigma)}\right|^{2-n} \qquad (A6)$$

This relation implies formula (32) for the angle $\frac{1}{2}(\wedge - \sigma)$. Equation (A5) may be written in the form

$$\frac{\zeta_1 + R_1\ e^{i\beta}}{\zeta_1 - R_1\ e^{-i\beta}} = \left(\frac{Z + Re^{i\sigma}}{Z - Re^{-i\sigma}}\right)^{2-n} \qquad (A7)$$

where R_1 and β are given by (25) and (26) and

$$\zeta_1 = \zeta + i(\sin\delta - R_1 \sin\beta)$$

Let $\zeta_1 = R_1\ e^{i\psi}$ be a point of the lower arc of the profile $\sqcap$ and $Z = Re^{i\wedge}$ the corresponding point of the circle $|Z| = R$. Substituting in (A7) and taking absolute values yields

$$\left|\frac{\cos\frac{1}{2}(\psi - \beta)}{\sin\frac{1}{2}(\psi + \beta)}\right| = \left|\frac{\cos\frac{1}{2}(\wedge - \sigma)}{\sin\frac{1}{2}(\wedge + \sigma)}\right|^{2-n} \qquad (A9)$$

This relation implies formula (33) for the angle $\frac{1}{2}(\wedge - \sigma)$.

In order to compute the speed of the incompressible flow around $\sqcap$, $|G'(\zeta)|$, at a point of the profile, note that

$$|G'(\zeta)| = \left|\frac{dG}{dZ}\right| \left|\frac{dZ}{d\zeta}\right| \tag{A10}$$

For $Z = Re^{i\wedge}$,

$$\left|\frac{dG}{dZ}\right| = 2\left[\sin\wedge + \sin\sigma\right] \tag{A11}$$

Furthermore, for a point $Z = Re^{i\wedge}$ corresponding to a point of the upper arc of $\sqcap$ (i.e., for $-\sigma < \wedge < \pi + \sigma$)

$$\left|\frac{dZ}{d\zeta}\right| = R\left|\frac{d\wedge}{d\Phi}\right| \tag{A12}$$

and by (A6)

$$\left|\frac{d\wedge}{d\Phi}\right| = \frac{\cos\delta}{(2-n)\cos\sigma}\left|\frac{\sin\wedge + \sin\sigma}{\sin\Phi + \sin\delta}\right| \tag{A13}$$

By (A10), (A11), (A12), and (A13)

$$|G'(e^{i\Phi})| = \frac{2\cos^2\delta}{(2-n)^2\cos^2\sigma}\,\frac{(\sin\wedge + \sin\sigma)^2}{|\sin\Phi + \sin\delta|}$$

This is equation (30).

Similarly, for a point $Z = Re^{i\wedge}$ corresponding to a point of the lower arc of $\sqcap$ (i.e., for $\sigma - \pi < \wedge < -\sigma$)

$$\left|\frac{dZ}{d\zeta}\right| = \frac{R}{R_1}\left|\frac{d\wedge}{d\psi}\right| \tag{A14}$$

By (A9)

$$\left|\frac{d\wedge}{d\psi}\right| = \frac{\cos\beta}{(2-n)\cos\sigma}\left|\frac{\sin\wedge + \sin\sigma}{\sin\psi + \sin\beta}\right| \tag{A15}$$

Thus, by (A10), (A11), (A14), and (A15)

$$\left|G'\left[R_1\,e^{i\psi} - i(\sin\delta - R_1\sin\beta)\right]\right| = \frac{2\cos^2\beta}{(2-n)^2\cos^2\sigma}\,\frac{(\sin\wedge + \sin\sigma)^2}{|\sin\psi + \sin\beta|}$$

This is formula (31).

TABLE II.- SOLUTIONS OF EQUATION (36)

n	1.022	1.024	1.026	1.028	1.030	1.032	1.034	1.036	1.038	1.040
ϵ	11.94°	16.45°	19.65°	22.15°	24.21°	25.95°	27.45°	28.78°	29.95°	31.01°
n	1.042	1.044	1.046	1.048	1.050	1.052	1.054	1.056	1.058	1.060
ϵ	31.96°	32.82°	33.62°	34.34°	35.01°	35.63°	36.21°	36.75°	37.25°	37.72°
n	1.062	1.064	1.066	1.068	1.070	1.072	1.074	1.076	1.078	1.080
ϵ	38.16°	38.57°	38.96°	39.33°	39.67°	40.00°	40.31°	40.60°	40.88°	41.15°

TABLE III

Radii of curvature of the profiles P

	δ	10°	10°	10°	11.436°	20.209°
	n	1.005	1.021	1.048	1.068	1.050
	α	9.50°	7.94°	5.42°	5°	15°
	M_∞	.1	.2	.3	.380	.324
	ϕ	R_P	R_P	R_P	R_P	R_P
Upper arc	90°	.998	.995	1.000	.988	.957
	80°	.998	.995	1.002	.991	.960
	70°	.999	.998	1.007	1.001	.969
	60°	1.000	1.003	1.019	1.016	.983
	50°	1.002	1.011	1.030	1.033	1.000
	40°	1.003	1.016	1.041	1.050	1.018
	30°	1.005	1.021	1.048	1.063	1.034
	20°	1.005	1.022	1.048	1.068	1.046
	10°	1.004	1.019	1.034	1.058	1.050
	0°	1.002	1.006	1.001	1.024	1.042
	-10°				.883	1.001
	-20°					.830
	ψ	R_P	R_P	R_P	R_P	R_P
Lower arc	0°	.987			.827	
	-10°	.999	.977	.972	.998	
	-20°	1.002	1.002	1.016	1.034	.960
	-30°	1.002	1.009	1.030	1.046	.991
	-40°	1.002	1.012	1.033	1.045	1.002
	-50°	1.002	1.011	1.030	1.037	1.004
	-60°	1.002	1.007	1.024	1.027	1.002
	-70°	1.001	1.005	1.018	1.017	.999
	-80°	1.001	1.003	1.013	1.010	.996
	-90°	1.001	1.003	1.012	1.008	.995

TABLE IV.- CRITICAL STREAM MACH NUMBER AS A FUNCTION

OF THE POSITION OF THE STAGNATION POINT

α	2°	4°	6°	8°	10°	12°	14°	16°	18°	20°
$M_{\infty,c}$	0.402	0.387	0.373	0.360	0.348	0.337	0.327	0.318	0.309	0.301
α	22°	24°	26°	28°	30°	32°	34°	36°	38°	40°
$M_{\infty,c}$	0.293	0.286	0.279	0.273	0.267	0.262	0.257	0.252	0.248	0.244
α	42°	44°	46°	48°	50°	52°	54°	56°	58°	60°
$M_{\infty,c}$	0.240	0.236	0.232	0.229	0.226	0.223	0.221	0.219	0.217	0.215

TABLE V

Ratio of the circulation of the compressible flow to the circulation of the incompressible flow.

α \ M_∞	.10	.15	.20	.25	.30	.35
1°	1.01489	1.03357	1.05984	1.09383	1.13570	1.18564
2°	1.01475	1.03325	1.05926	1.09290	1.13431	1.18366
3°	1.01461	1.03293	1.05869	1.09198	1.13293	1.18169
4°	1.01448	1.03262	1.05812	1.09106	1.13156	1.17975
5°	1.01434	1.03231	1.05756	1.09015	1.13020	1.17782
7½°	1.01401	1.03155	1.05616	1.08790	1.12684	
10°	1.01367	1.03079	1.05477	1.08567	1.12352	
15°	1.01302	1.02928	1.05203	1.08127	1.11696	
20°	1.01236	1.02777	1.04930	1.07689	1.11044	
30°	1.01102	1.02472	1.04377	1.06802		
40°	1.00962	1.02153	1.03798			
45°	1.00888	1.01984	1.03494			

TABLE VI

Lift Coefficient

α \ M_∞	.10	.15	.20	.25	.30	.35
1°	.4452	.4534	.4649	.4798	.4982	.5201
2°	.8901	.9063	.9291	.9586	.9949	1.0382
3°	1.3346	1.3587	1.3925	1.4363	1.4902	1.5543
4°	1.7786	1.8104	1.8551	1.9128	1.9838	2.0683
5°	2.2219	2.2612	2.3165	2.3879	2.4757	2.5800
7½°	3.3264	3.3840	3.4647	3.5688	3.6966	
10°	4.4239	4.4986	4.6033	4.7382	4.9033	
15°	6.5895	6.6953	6.8433	7.0335	7.2656	
20°	8.7021	8.8347	9.0197	9.2568	9.5453	
30°	12.7049	12.8770	13.1164	13.4212		
40°	16.3104	16.5028	16.7686			
45°	17.9293	18.1241	18.3924			

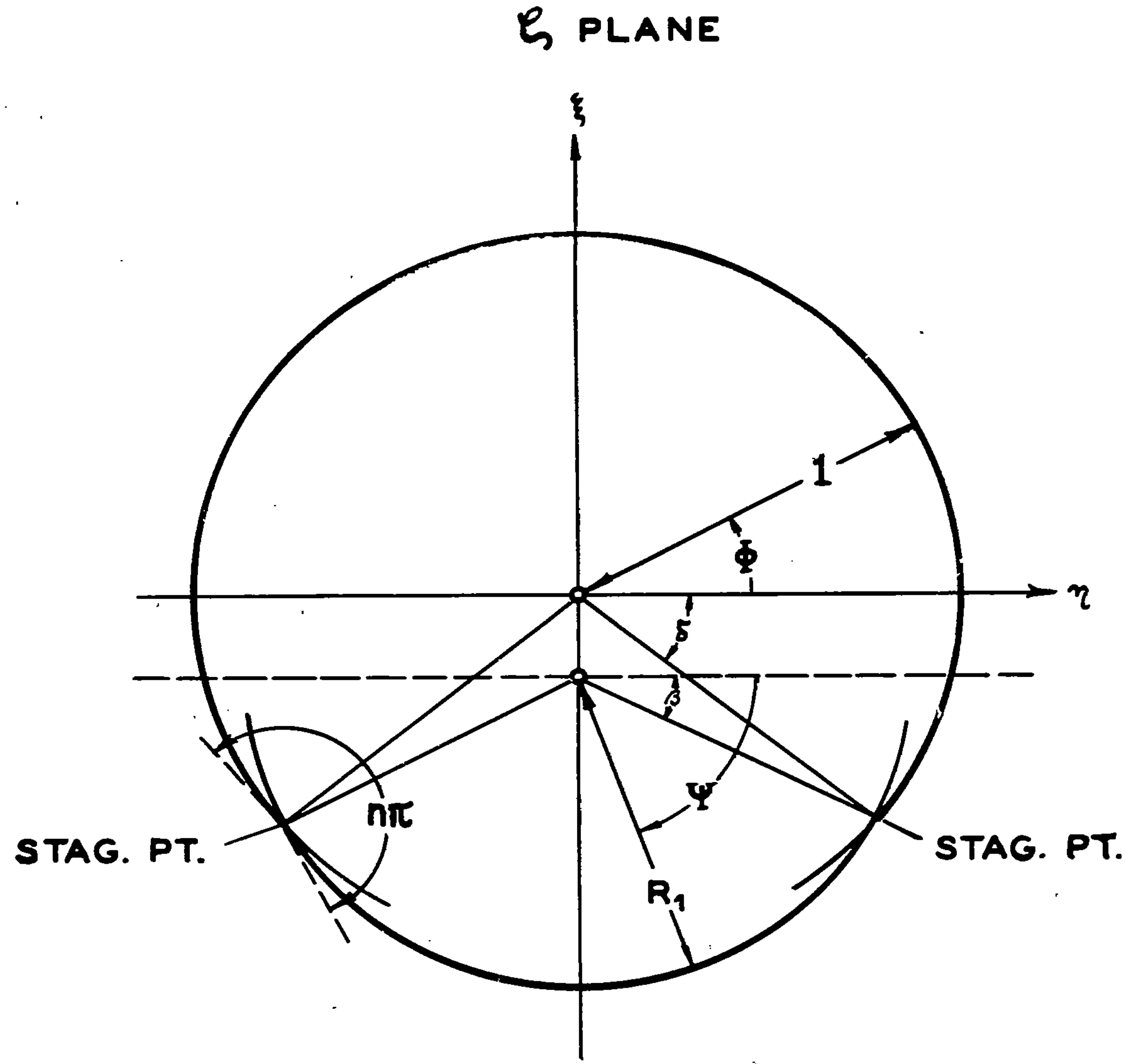


FIG. I

THE AUXILIARY PROFILE Π

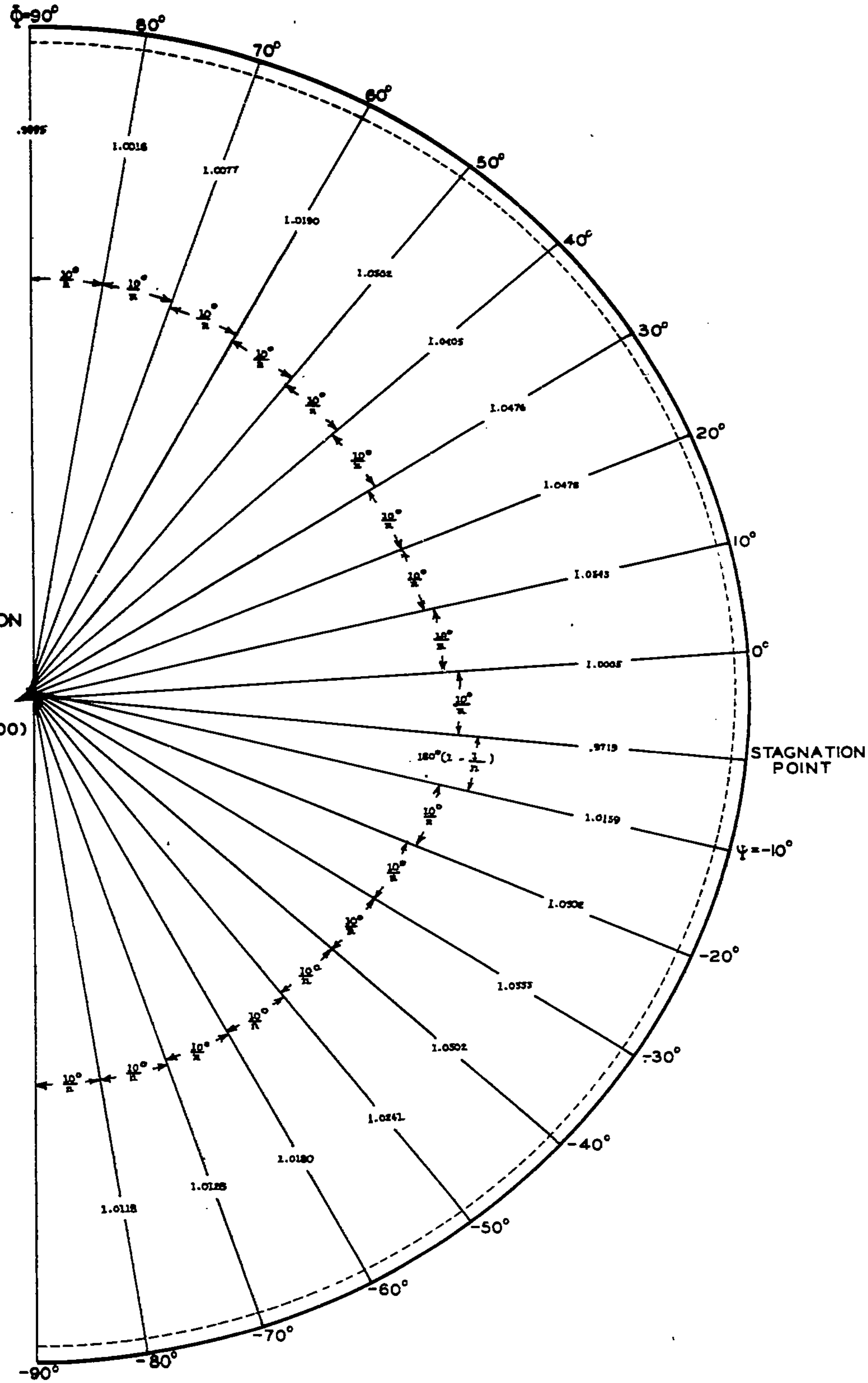
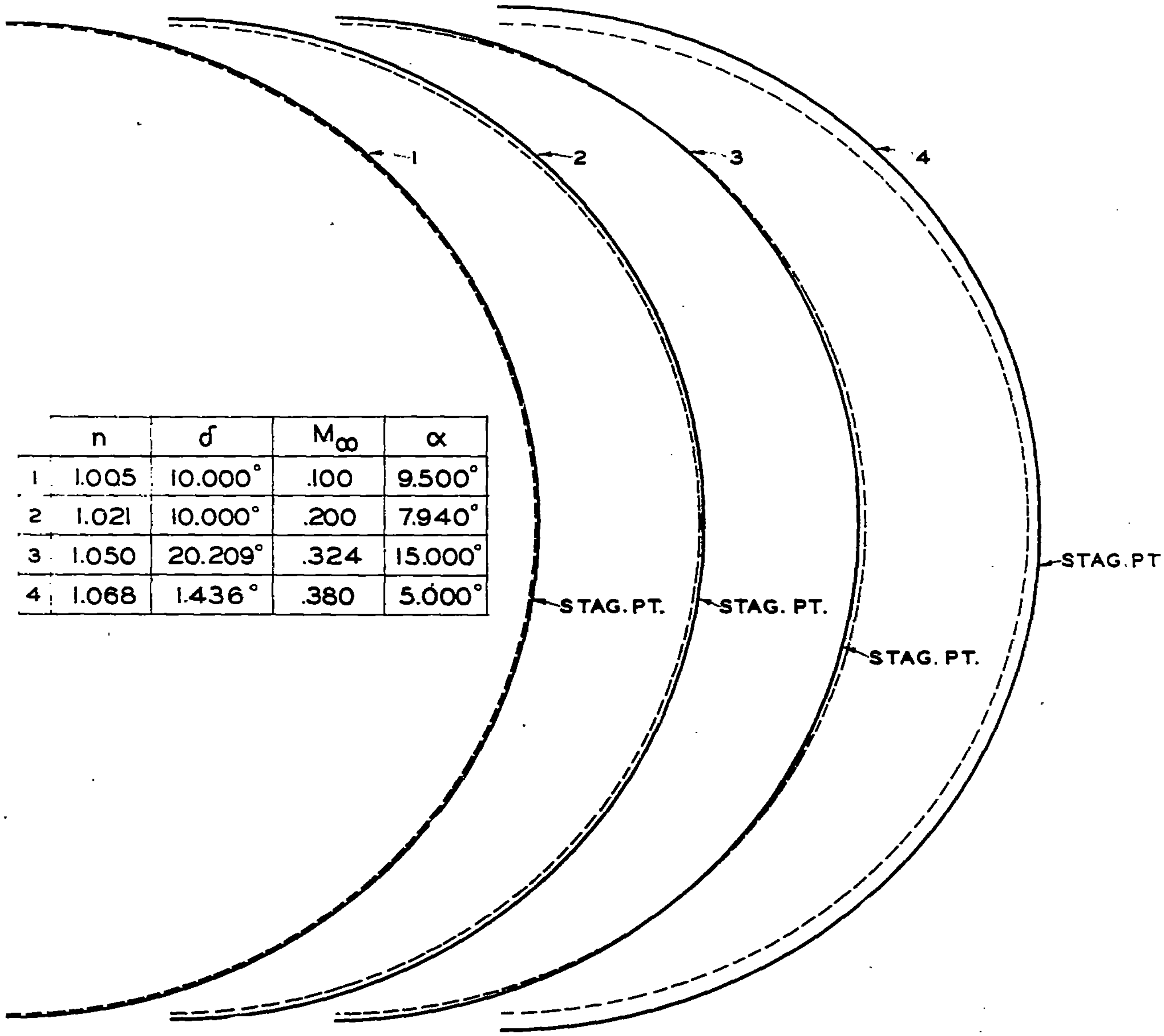


FIG 2
GRAPHICAL CONSTRUCTION
OF THE PROFILE P

(δ=10° n=1.048 α=5.423° M_∞=300)

----UNIT CIRCLE



	n	δ	M_∞	α
1	1.005	10.000°	.100	9.500°
2	1.021	10.000°	.200	7.940°
3	1.050	20.209°	.324	15.000°
4	1.068	1.436°	.380	5.000°

FIG. 3
PROFILES P
---- UNIT CIRCLE

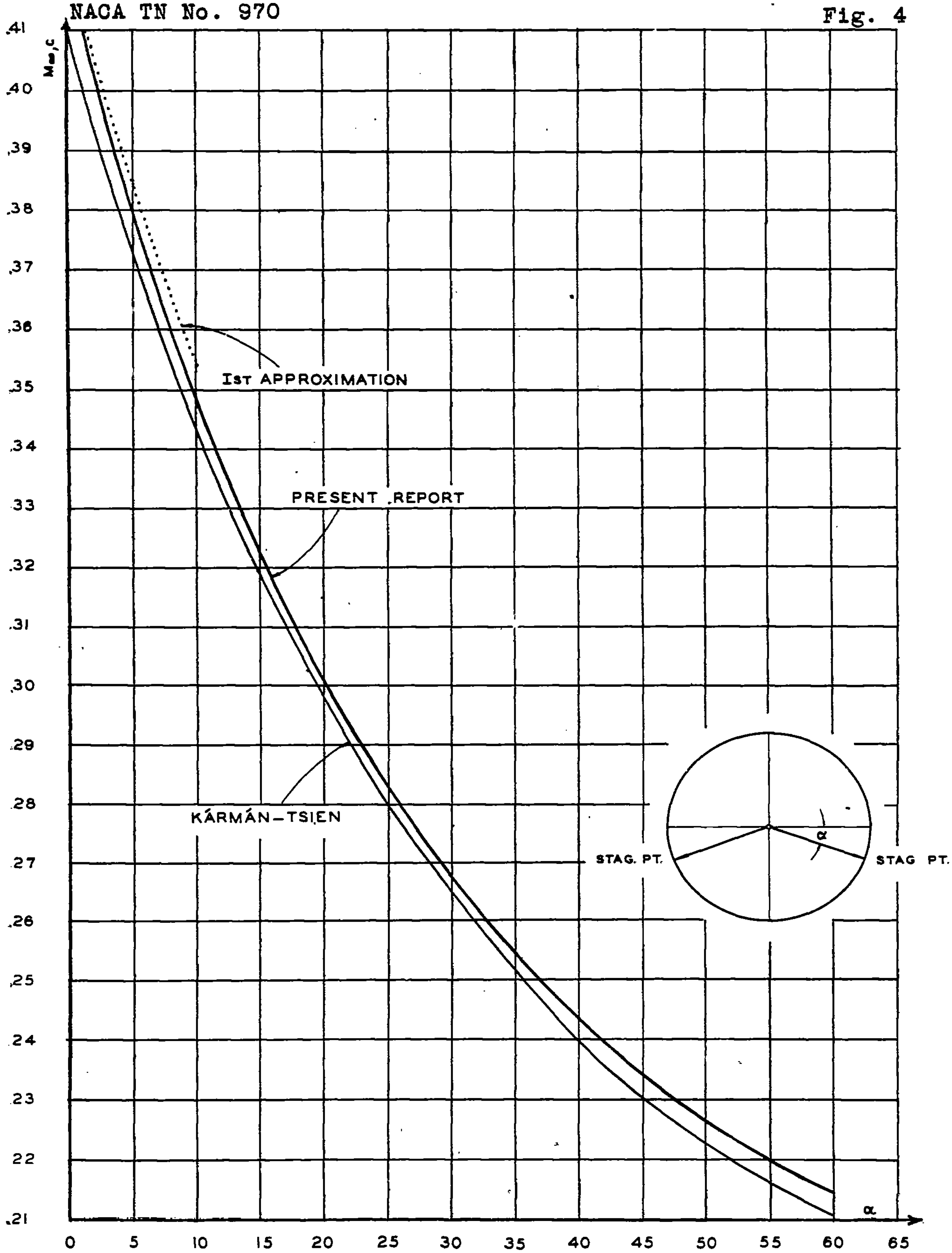


Figure 4.- Critical stream Mach number as a function of the position of the stagnation point.

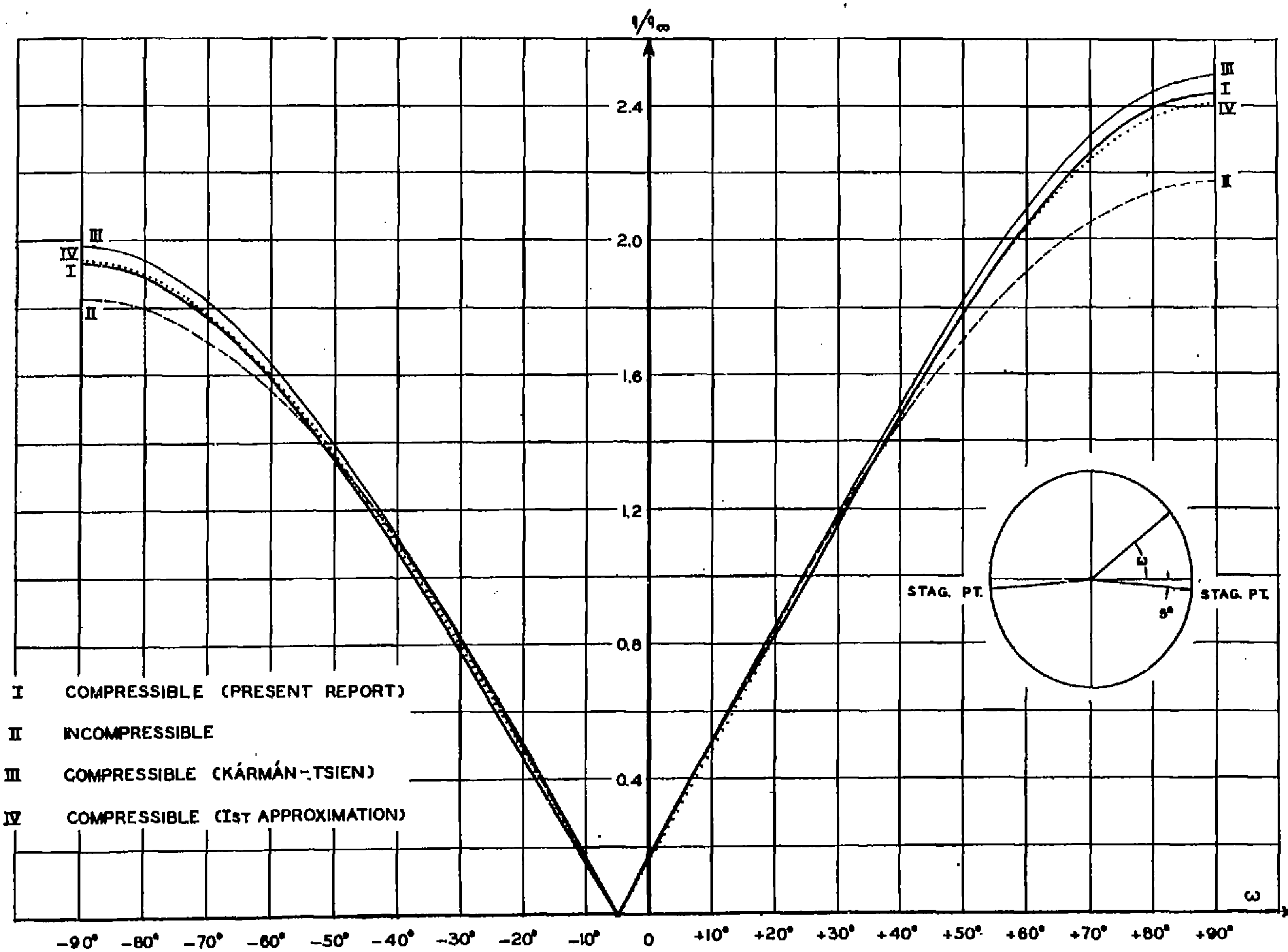


Figure 5a.- Velocity distribution. $\alpha = 5^0$, $M_\infty = .380$.

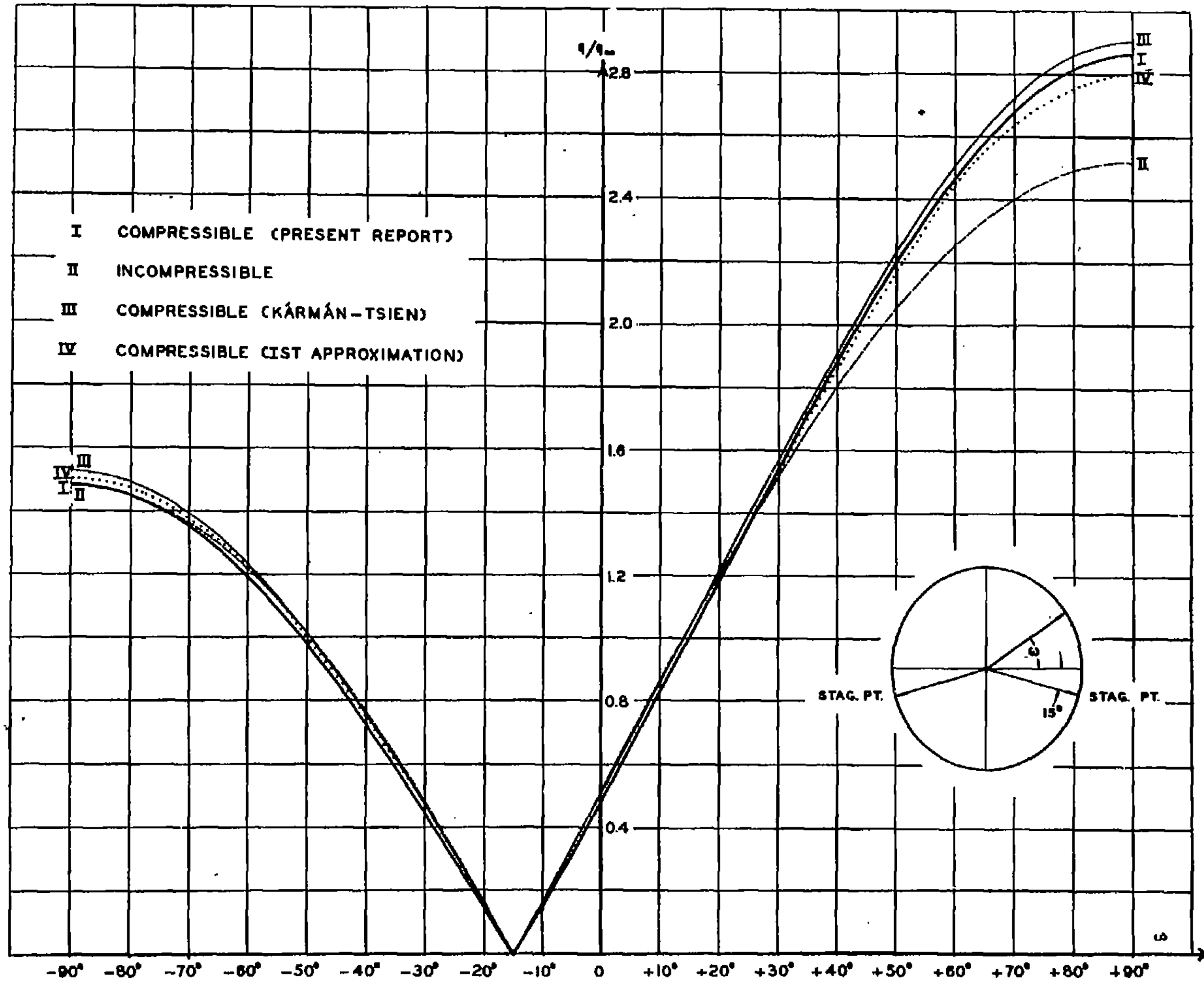


Figure 5b.- Velocity distribution. $\alpha = 15^{\circ}$, $M = .324$.

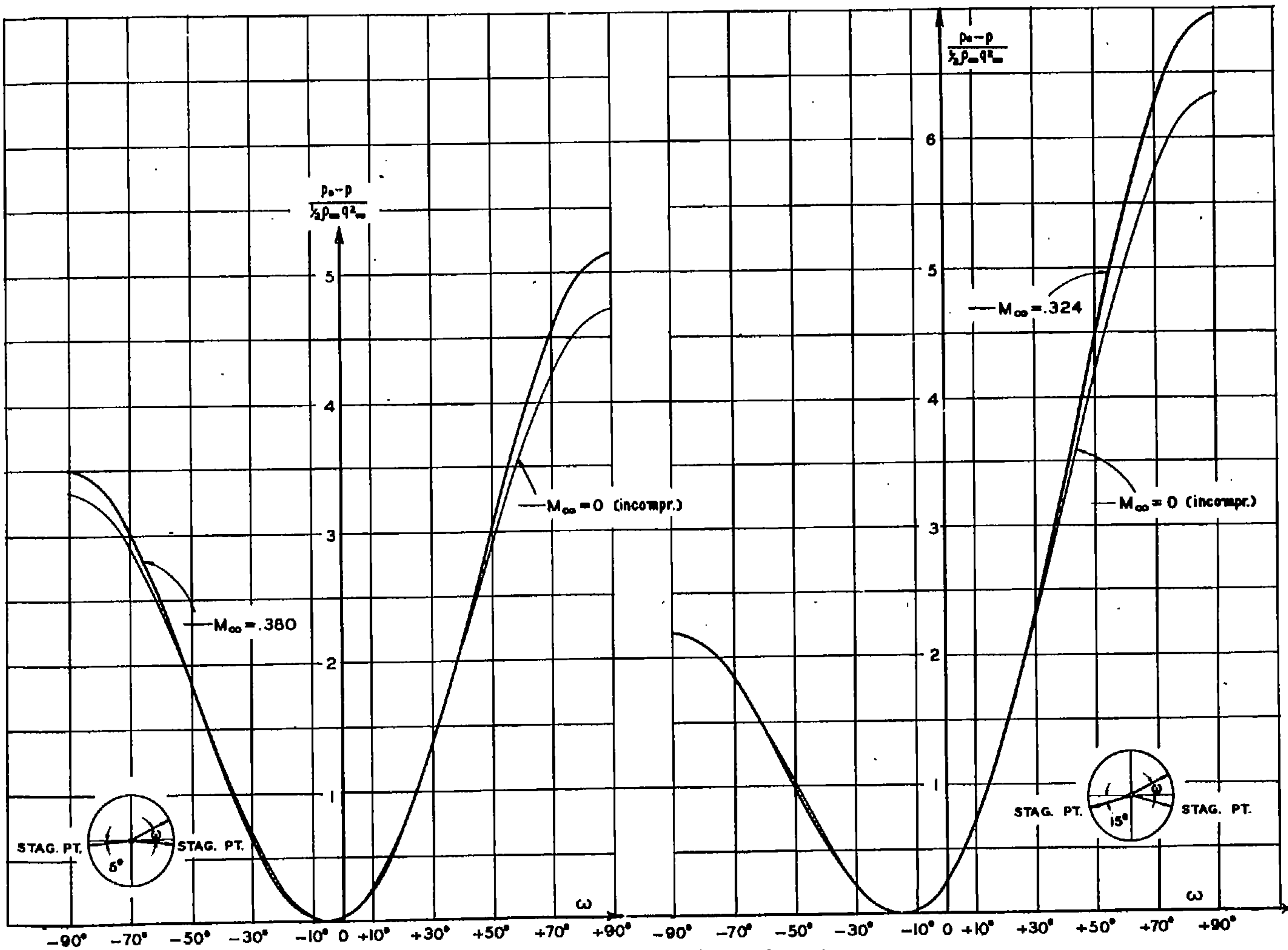


Figure 6.- Pressure distribution.

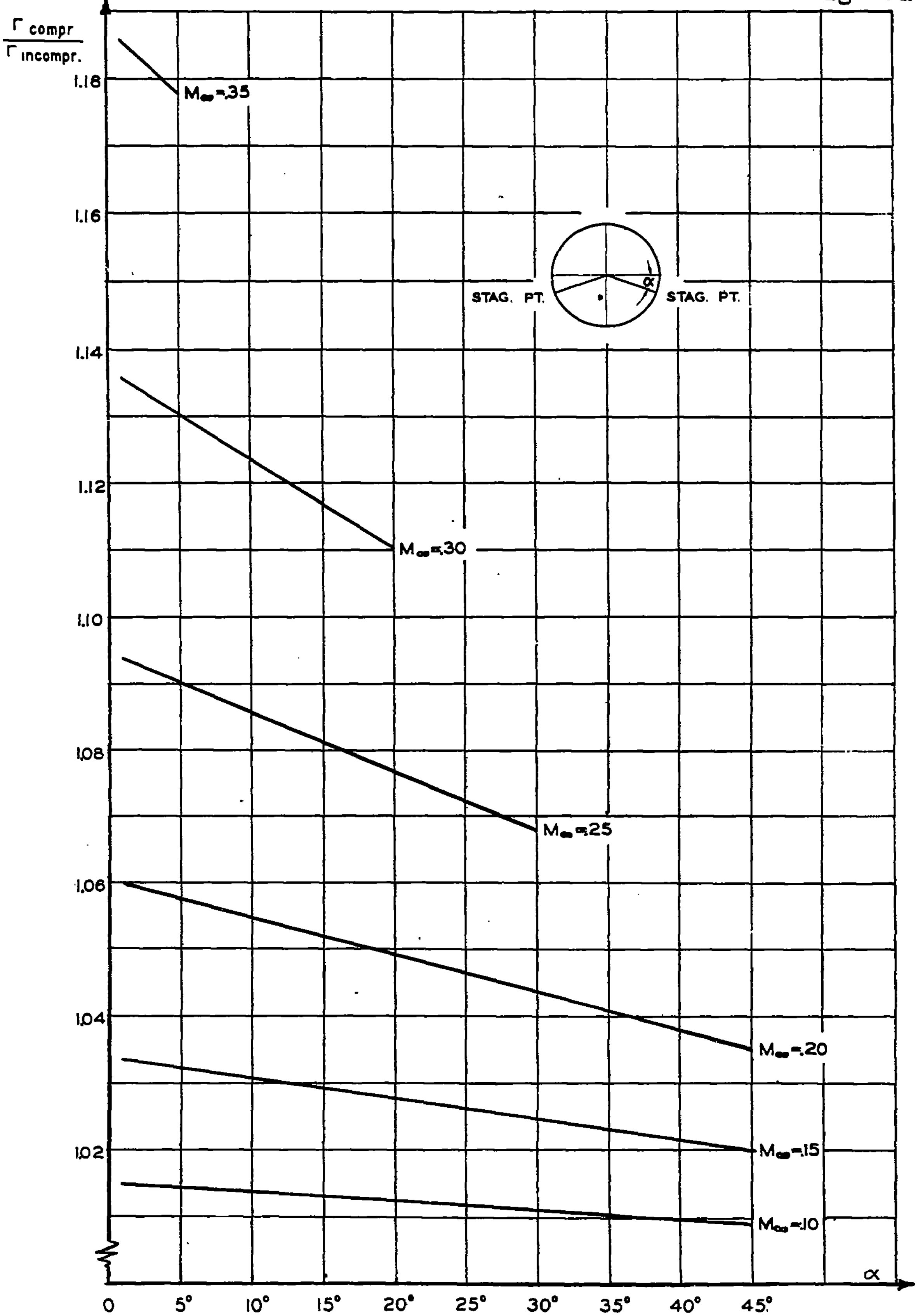


Figure 7a.- Circulation as a function of the position of the stagnation point.

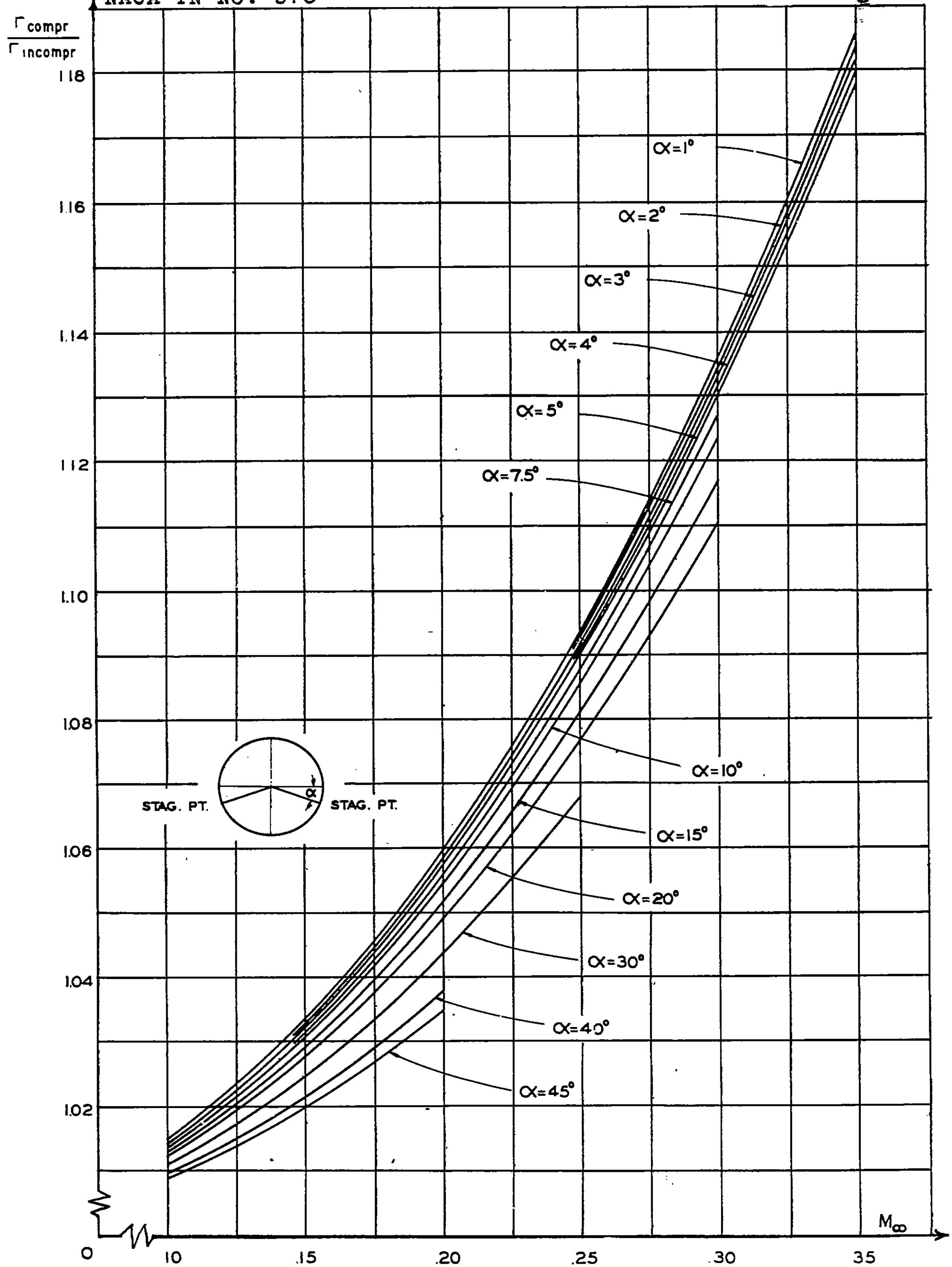


Figure 7b.- Circulation as a function of the stream Mach number.

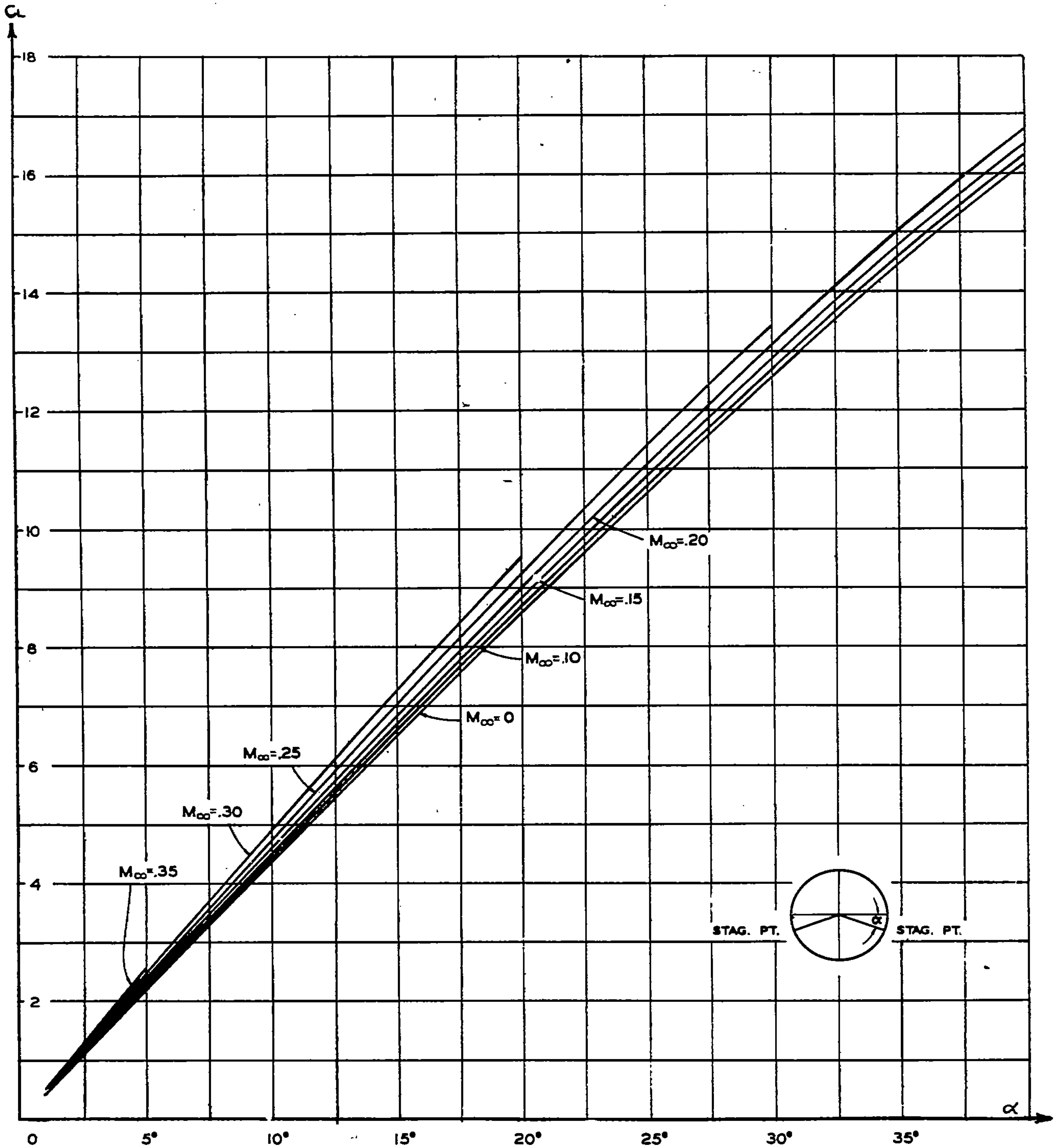


Figure 8a.- Lift coefficient as a function of the position of the stagnation point.

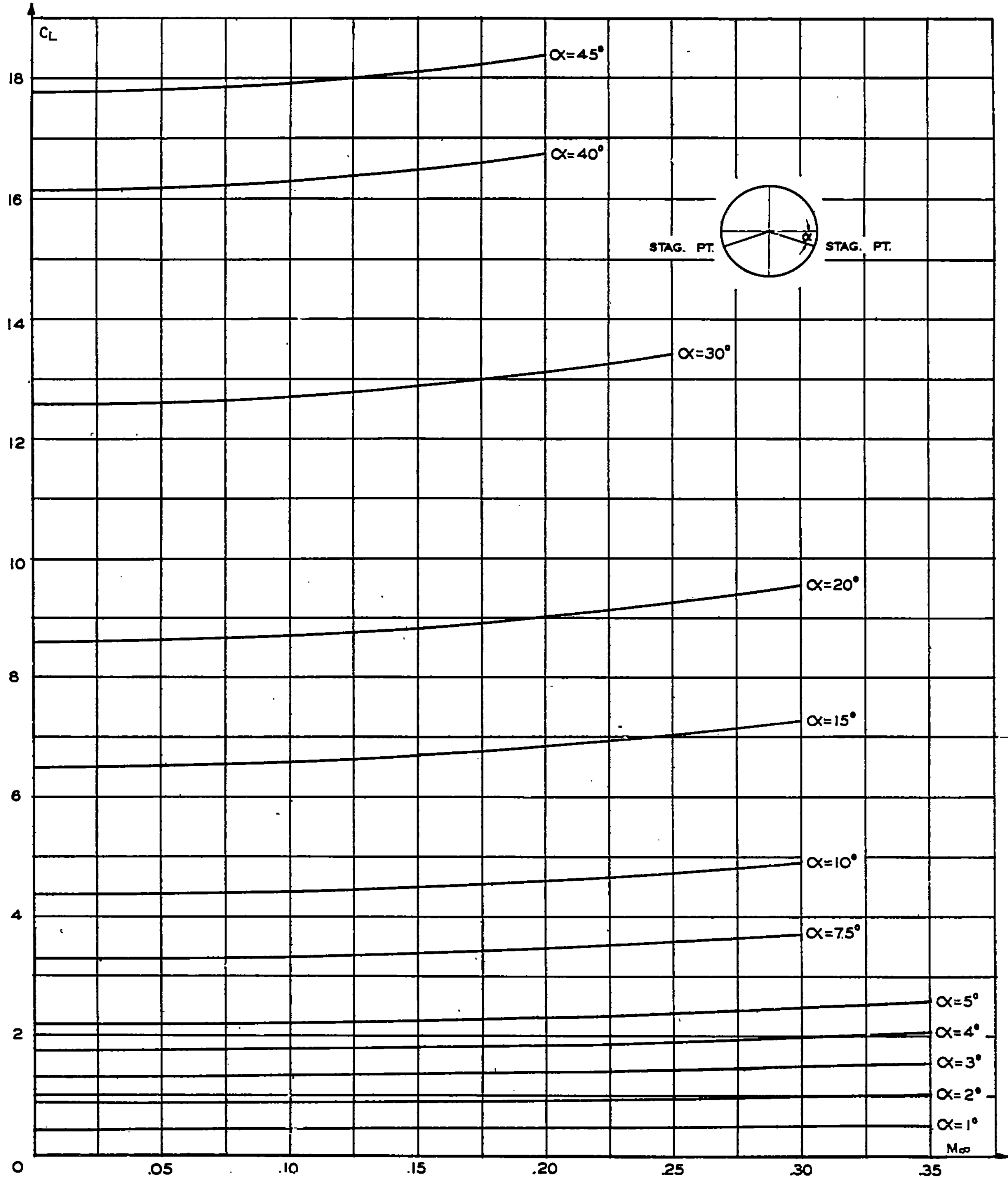


Figure 8b.- Lift coefficient as a function of the stream Mach number.

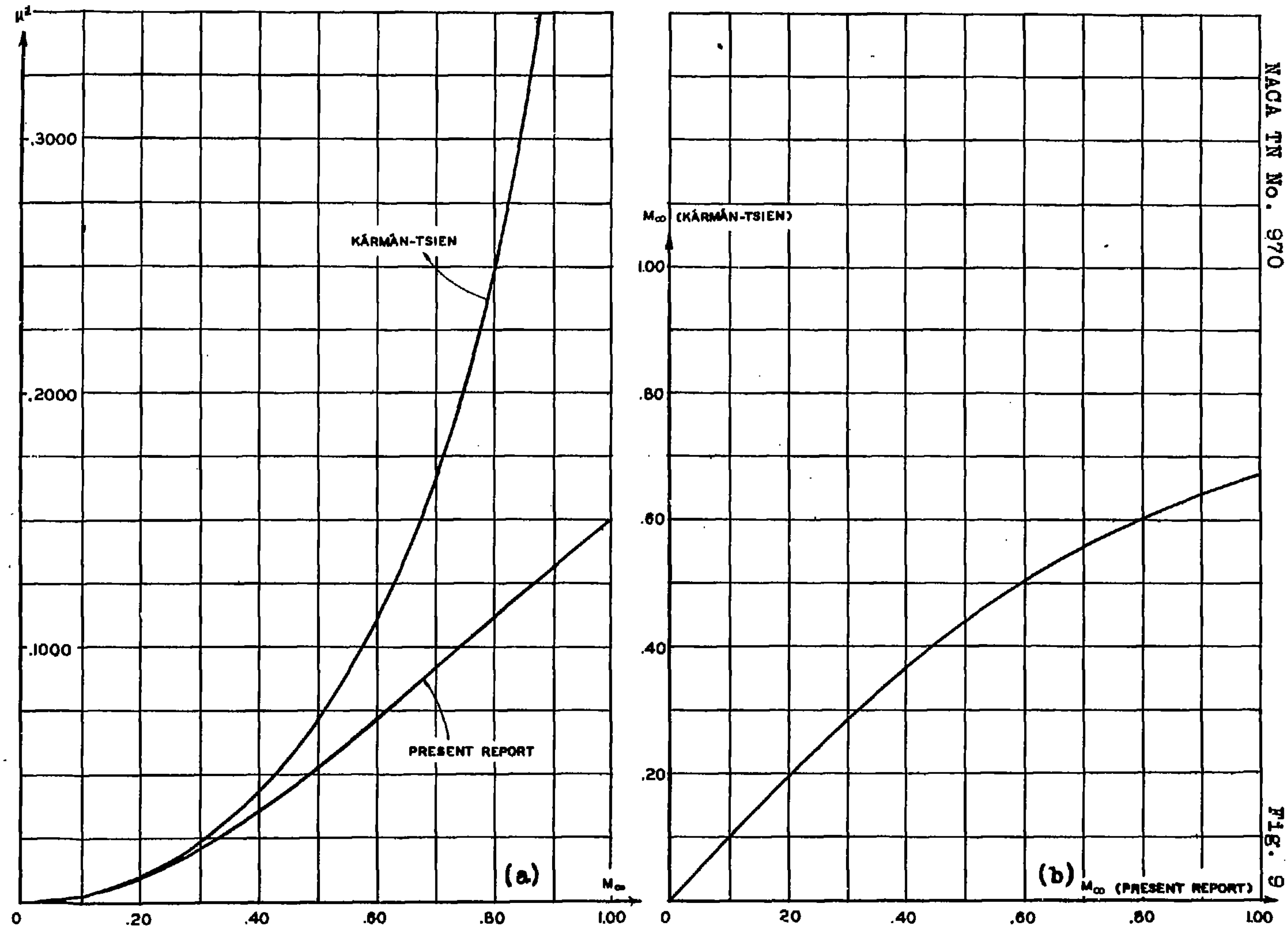

μ²
.3000
.2000
.1000
KÁRMÁN-TSIEN
PRESENT REPORT
(a)
M∞
0
.20
.40
.60
.80
1.00
M∞ (KÁRMÁN-TSIEN)
1.00
.80
.60
.40
.20
(b)
M∞ (PRESENT REPORT)
0
.20
.40
.60
.80
1.00

Figure 9